Linearized theory for entire solutions of a singular Liouvillle equation

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July 21, 2010

Abstract

We discuss invertibility properties for entire finite-energy solutions of the regularized version of a singular Liouvillle equation.

1 The linear theory

In this short note, we want to show the following result

Theorem 1.1. *Let $c \in \mathbb{C}$ and $N \in \mathbb{N}$. The kernel of the operator*

$$L: \phi \to L(\phi) := \Delta\phi + \frac{8(N+1)^2|z|^{2N}}{(1+|z^{N+1}-c|^2)^2}\phi$$

in $L^\infty(\mathbb{R}^2)$ has the simple form

$$\{\phi \in L^\infty(\mathbb{R}^2) : L(\phi) = 0\} = Span\, \{\frac{1-|z^{N+1}-c|^2}{1+|z^{N+1}-c|^2}, \frac{Re\,(z^{N+1}-c)}{1+|z^{N+1}-c|^2}, \frac{Im\,(z^{N+1}-c)}{1+|z^{N+1}-c|^2}\},$$

where z^{N+1} denotes the complex $(N+1)-$power.

The case $N = 0$ was already known (see [1]), based on a Fourier expansion of the function ϕ. The aim here is to adapt the argument to the more difficult case $N \in \mathbb{N}$.

Proof: Let us recall the Liouville formula: given a holomorphic function f on $\mathbb{C}$, the function

$$\ln \frac{8|f'(z)|^2}{(1+|f(z)|^2)^2} \tag{1}$$

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solves the equation $\Delta U + e^U = 0$ in the set $\{z \in \mathbb{C} \ / \ f'(z) \neq 0\}$. If now f' has a zero at the origin of multiplicity N, the function

$$\ln \frac{8|f'(z)|^2}{(1+|f(z)|^2)^2} - \ln |z|^{2N} \tag{2}$$

solves the equation $\Delta U + |z|^{2N} e^U = 0$ in the set $\{z \in \mathbb{C}\setminus\{0\} \ / \ f'(z) \neq 0\}$. The choice $f(z) = z^{N+1}(1+\tau z^k) - c$, $k \geq 0$, leads to a family

$$U_{\tau,k}(z) = \ln \frac{8(N+1)^2 |1 + \tau \frac{N+1+k}{N+1} z^k|^2}{(1+|z^{N+1}(1+\tau z^k) - c|^2)^2}, \ \ \tau \in \mathbb{C},$$

of solutions of $\Delta U + |z|^{2N} e^U = 0$ in $\mathbb{C} \setminus \{z \in \mathbb{C} : 1 + \tau \frac{N+1+k}{N+1} z^k = 0\}$. The derivative of $U_{\tau,k}$ in τ at $\tau = 0$:

$$\phi_k := \partial_\tau U_{\tau,k}\Big|_{\tau=0} = z^k \left(\frac{N+1+k}{N+1} - 2 \frac{z^{N+1}\overline{z^{N+1} - c}}{1+|z^{N+1} - c|^2} \right)$$

solves $L(\phi_k) = 0$ in $\mathbb{C}$, and in particular, $\phi_0 = \mathrm{Re}\ \phi_0$ and $\phi_k^1 = \frac{N+1}{N+1+k} \mathrm{Re}\ \phi_k$, $\phi_k^2 = \frac{N+1}{N+1+k} \mathrm{Im}\ \phi_k$ are real solutions for $k \geq 1$. We want to show that every solution ϕ of $L(\phi) = 0$ is a linear combination of ϕ_0 and ϕ_k^i, $k \geq 1$ and $i = 1, 2$:

$$\phi = a_0 \phi_0 + \sum_{k \geq 1} \left(a_k \phi_k^1 + b_k^1 \phi_k^2 \right). \tag{3}$$

The key idea is that, for ρ small, the functions $\phi_0(\rho e^{i\theta})$ and $\frac{1}{\rho^k} \phi_k^i(\rho e^{i\theta})$, $k \geq 1$ and $i = 1, 2$, are very close to the real Fourier basis 1, $\cos(k\theta)$ and $\sin(k\theta)$ with $k \geq 1$, and then form a complete set in $L^2(\partial B_\rho(0))$. Indeed, by a standard integration by parts we can compute for $k \geq 1$

$$\int_{S^1} \phi_0(\rho e^{i\theta}) d\theta = 1 + O(\rho^{N+1}), \quad \int_{S^1} \cos(k\theta) \phi_0(\rho e^{i\theta}) d\theta = O(\frac{\rho^{N+1}}{k+1}), \quad \int_{S^1} \sin(k\theta) \phi_0(\rho e^{i\theta}) d\theta = O(\frac{\rho^{N+1}}{k+1})$$

and

$$\int_{S^1} \cos(k\theta) \phi_j^1(\rho e^{i\theta}) d\theta = \pi \delta_{kj} + O(\frac{\rho^{N+1}}{(k+1)(j+1)}), \quad \int_{S^1} \sin(k\theta) \phi_j^1(\rho e^{i\theta}) d\theta = O(\frac{\rho^{N+1}}{(k+1)(j+1)})$$

and

$$\int_{S^1} \cos(k\theta) \phi_j^2(\rho e^{i\theta}) d\theta = O(\frac{\rho^{N+1}}{(k+1)(j+1)}), \quad \int_{S^1} \sin(k\theta) \phi_j^2(\rho e^{i\theta}) d\theta = \pi \delta_{kj} + O(\frac{\rho^{N+1}}{(k+1)(j+1)}).$$

Letting $\psi \in L^2(\partial B_\rho(0))$ in the form

$$\psi(\rho e^{i\theta}) = c_0 + \sum_{k \geq 1} \left(c_k \cos(k\theta) + d_k \sin(k\theta) \right),$$

we can compute

$$\tilde{c}_0 := \int_{S^1} \psi(\rho e^{i\theta}) \phi_0(\rho e^{i\theta}) d\theta = c_0 + \rho^{N+1} O\left(\sum_{k \geq 0} \frac{|c_k|}{k+1} + \sum_{k \geq 1} \frac{|d_k|}{k+1} \right)$$

and

$$\tilde{c}_j := \int_{S^1} \psi(\rho e^{i\theta}) \phi_j^1(\rho e^{i\theta}) d\theta = c_j + \frac{\rho^{N+1}}{j+1} O\left(\sum_{k \geq 0} \frac{|c_k|}{k+1} + \sum_{k \geq 1} \frac{|d_k|}{k+1} \right)$$

$$\tilde{d}_j := \int_{S^1} \psi(\rho e^{i\theta})\phi_j^2(\rho e^{i\theta})d\theta = d_j + \frac{\rho^{N+1}}{j+1}\left(\sum_{k\geq 0}\frac{|c_k|}{k+1} + \sum_{k\geq 1}\frac{|d_k|}{k+1}\right).$$

We consider the operator

$$T : (c_0, c_1, d_1, \dots) \in l_2 \ \to\ (\tilde{c}_0, \tilde{c}_1, \tilde{;}d_1, \dots) \in l^2.$$

We have shown so far that

$$\|T - \mathrm{Id}\| \leq C\rho^{N+1}\left(\sum_{j\geq 0}\frac{1}{(j+1)^2}\right).$$

In conclusion, for ρ small we have that T is an invertible operator, and by injectivity we then deduce that, if $\psi \in L^2(\partial B_\rho(0))$ is so that

$$\int_{S^1} \psi(\rho e^{i\theta})\phi_0(\rho e^{i\theta})d\theta = \int_{S^1} \psi(\rho e^{i\theta})\phi_k^j(\rho e^{i\theta})d\theta = 0 \quad \forall : k\geq 1,\ j = 1, 2,$$

then its Fourier coefficients c_j vanish and $\psi = 0$. This means that, for ρ small, the space $L^2(\partial B_\rho(0))$ coincides with the closure in L^2-norm of

$$\text{Span }\{\phi_0,\ \phi_k^j :\ k\geq 1,\ j = 1, 2\}.$$

In particular, every solution $\phi \in L^\infty(\mathbb{C})$ can be written on $\partial B_\rho(0)$, ρ small, as

$$\phi(\rho e^{i\theta}) = a_0\phi_0(\rho e^{i\theta}) + \sum_{k\geq 1}\left(a_k\phi_k^1(\rho e^{i\theta}) + b_k\phi_k^2(\rho e^{i\theta})\right),$$

for suitable a_j and b_j. By regularity theory $\phi \in C^\infty(\mathbb{C})$, and then $\phi\mid_{\partial B_\rho(0)} \in C^\infty(\partial B_\rho(0))$. Arguing as for the Fourier coefficients, it is easily seen that a_k and b_k tend to zero as $k \to +\infty$ faster than any power of k. In particular, the function

$$\hat{\phi}(z) = a_0\phi_0(z) + \sum_{k\geq 1}[a_k\phi_k^1(z) + b_k\phi_k^2(z)]$$

is well defined, is in $C^\infty(\mathbb{C})$ and satisfies $L(\hat{\phi}) = 0$ in $\mathbb{C}$. Since $\phi = \hat{\phi}$ on $\partial B_\rho(0)$ and $\delta = \phi - \hat{\phi}$ satisfies $L(\delta) = 0$ in $\mathbb{C}$, an integration by parts yields to

$$\int_{B_\rho(0)} |\nabla\delta|^2 = \int_{B_\rho(0)} V\delta^2 - \int_{B_\rho(0)} L(\delta)\delta = \int_{B_\rho(0)} V\delta^2 \leq C\rho^{2N}\int_{B_\rho(0)}\delta^2,$$

where $V(z) = \frac{8(N+1)^2|z|^{2N}}{(1+|z^{N+1}-c|^2)^2}$. As soon as $C\rho^{2N} < \lambda_1(B_\rho(0))$ (λ_1 being the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions), we get that necessarily $\delta = 0$ in $B_\rho(0) = 0$. Then, for ρ small we get that $\delta = 0$ in $B_\rho(0)$, and by the strong maximum principle $\delta = 0$ in $\mathbb{C}$. So we have shown that

$$\phi(z) = a_0\phi_0(z) + \sum_{k\geq 1}[a_k\phi_k^1(z) + b_k\phi_k^2(z)]$$

in $\mathbb{C}$. Let us look now at the behavior of $\phi(z)$ as $|z| \to +\infty$. Since the only bounded components in $\phi(z)$ are ϕ_0 and ϕ_{N+1}^1, ϕ_{N+1}^2, we expect that $a_k = b_k = 0$ for $k \neq 0, N+1$. Also in this case we will use that the components of ϕ are very close to the Fourier basis as $|z| \to +\infty$.
Indeed, observe that

$$\frac{z^{N+1}\overline{z^{N+1}-c}}{1+|z^{N+1}-c|^2} = 1 + O(\frac{1}{|z|^{N+1}}) \qquad \text{as } |z| \to +\infty,$$

and then

$$\phi_k(z) = z^k \left(\frac{k-N-1}{N+1} + O(\frac{1}{|z|^{N+1}}) \right)$$

at infinity. So we have that

$$\phi_0(z) = -1 + O(\frac{1}{|z|^{N+1}})\,, \quad \phi_k^(z) = \frac{k-N-1}{N+1+k}|z|^k \cos(k\theta)(1 + O(\frac{1}{|z|^{N+1}})$$
$$\phi_k^2(z) = \frac{k-N-1}{N+1+k}|z|^k \sin(k\theta)(1 + O(\frac{1}{k|z|^{N+1}})).$$

Let us now compute by Cauchy-Schwartz inequality

$$\begin{aligned}
\frac{1}{R}\int_{\partial B_R} |\phi|^2 &= \pi \left(\sum_{k\geq 0} \frac{(k-N-1)^2}{(N+1+k)^2} R^{2k}|a_k|^2 + \sum_{k\geq 1} \frac{(k-N-1)^2}{(N+1+k)^2} R^{2k}|b_k|^2 \right) \\
&\quad + o\left(\sum_{k,j} R^{k+j}(|a_k||a_j| + |b_k||b_j| + |a_k||b_j|) \right) \\
&= \pi(1+o(1)) \left(\sum_{k\geq 0} \frac{(k-N-1)^2}{(N+1+k)^2} R^{2k}|a_k|^2 + \sum_{k\geq 1} \frac{(k-N-1)^2}{(N+1+k)^2} R^{2k}|b_k|^2 \right)
\end{aligned}$$

as $R \to +\infty$. Since $\phi \in L^\infty(\mathbb{C})$, we have that $\frac{1}{R}\int_{\partial B_R} |\phi|^2$ is bounded in R, and then

$$\sum_{k\geq 0} \frac{(k-N-1)^2}{(N+1+k)^2} R^{2k}|a_k|^2 + \sum_{k\geq 1} \frac{(k-N-1)^2}{(N+1+k)^2} R^{2k}|b_k|^2$$

is bounded in R. Then $a_k = 0$ and $b_k = 0$ for $k \geq 1$ unless $k = N+1$. For a solution $\phi \in L^\infty(\mathbb{C})$ of $L(\phi) = 0$ we have then shown that

$$\phi(z) = a_0\phi_0(z) + a_{N+1}\phi^1_{N+1}(z) + b_{N+1}\phi^2_{N+1}(z).$$

To conclude, we need simply to rewrite ϕ_0 and ϕ_{N+1} in a more expressive way. We have that

$$\phi_0(z) = 1 - 2\frac{z^{N+1}\overline{z^{N+1}-c}}{1+|z^{N+1}-c|^2} = \frac{1-|z^{N+1}-c|^2}{1+|z^{N+1}-c|^2} - 2c\frac{\overline{z^{N+1}-c}}{1+|z^{N+1}-c|^2}$$

and

$$\phi_{N+1}(z) = 2z^{N+1}\left(1 - \frac{z^{N+1}\overline{z^{N+1}-c}}{1+|z^{N+1}-c|^2}\right) = 2c\frac{1-|z^{N+1}-c|^2}{1+|z^{N+1}-c|^2} + 2\frac{z^{N+1}-c}{1+|z^{N+1}-c|^2} - 2c^2\frac{\overline{z^{N+1}-c}}{1+|z^{N+1}-c|^2}.$$

In real form we can then write that

$$\phi_0(z) = \frac{1-|z^{N+1}-c|^2}{1+|z^{N+1}-c|^2} - 2c_1\mathrm{Re}\ \frac{z^{N+1}-c}{1+|z^{N+1}-c|^2} - 2c_2\mathrm{Im}\ \frac{z^{N+1}-c}{1+|z^{N+1}-c|^2}$$

and

$$\phi^1_{N+1}(z) = c_1\frac{1-|z^{N+1}-c|^2}{1+|z^{N+1}-c|^2} + (1-c_1^2+c_2^2)\mathrm{Re}\ \frac{z^{N+1}-c}{1+|z^{N+1}-c|^2} - 2c_1c_2\mathrm{Im}\ \frac{z^{N+1}-c}{1+|z^{N+1}-c|^2}$$

and

$$\phi^2_{N+1}(z) = c_2 \frac{1-|z^{N+1}-c|^2}{1+|z^{N+1}-c|^2} - 2c_1c_2 \mathrm{Re}\, \frac{z^{N+1}-c}{1+|z^{N+1}-c|^2} + (1+c_1^2-c_2^2)\mathrm{Im}\, \frac{z^{N+1}-c}{1+|z^{N+1}-c|^2},$$

where $c = c_1 + ic_2$. In conclusion, the function ϕ can be written as

$$\begin{aligned} \phi \quad = \quad & (a_0 + a_{N+1}c_1 + b_{N+1}c_2)\frac{1-|z^{N+1}-c|^2}{1+|z^{N+1}-c|^2} + [-2a_0c_1 + a_{N+1}(1-c_1^2+c_2^2) - 2b_{N+1}c_1c_2]\mathrm{Re}\, \frac{z^{N+1}-c}{1+|z^{N+1}-c|^2} \\ & + [-2a_0c_2 - 2a_{N+1}c_1c_2 + b_{N+1}(1+c_1^2-c_2^2)]\mathrm{Im}\, \frac{z^{N+1}-c}{1+|z^{N+1}-c|^2} \end{aligned}$$

and the Theorem is established. ■

References

[1] S. Baraket, F. Pacard, Construction of singular limits for a semilinear elliptic equation in dimension 2, *Calc. Var. P.D.E.* **6** (1998), 1-38.